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On the complexity of H-colouring planar graphs☆

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ABSTRACT

We show that if H is an odd-cycle, or any non-bipartite graph of girth 5 and maximum degree at most 3, then planar H-COL is NP-complete.

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1. Introduction

Given a graph H, the problem of whether or not another graph G admits an H-colouring, i.e., an edge-preserving vertex map to H, is referred to as H-COL. In [7], it was shown that H-COL is NP-complete if H contains an odd cycle, and polynomial-time solvable otherwise. (In this note, all graphs are assumed to be finite, symmetric, and irreflexive.)

One can ask whether the H-COL problem remains NP-complete when the instances G are given certain restrictions. A good reference for such problems is [6]. In this paper we focus on restricting to planar graphs G. We refer to the problem of H-COL, restricted to planar instances, as *planar H-COL*. In [3], it is shown that planar K_3-COL is NP-complete. At the same time, the four-colour theorem implies that planar H-COL is trivial for any graph H containing a K_4.

There are several examples of graphs for which planar H-COL is polynomial time solvable. For example, in [9], it is shown that a planar graph G maps to the Clebsch graph C if and only if it is K_3-free. This implies that planar C-COL is polynomial time solvable. This result has recently been extended in [10] to show that, in particular, for any planar graph F there is a graph U containing no homomorphic image of F, for which there is a U-colouring for any F-free planar graph G. Planar U-COL is polynomial time solvable for any such graph U.

However, it seems to be a difficult problem to determine for which H, planar H-COL is NP-complete. Indeed it has recently been shown [5] that there exists an infinite family of graphs $\{H\}_{i=1}^{\infty}$ such that $H_i \to H_{i-1}$ for all $i > 1$, and such that planar H_i-COL is NP-complete when i is odd, and polynomial time solvable when i is even.

Restriction to planar instances has been considered in other homomorphism problems. In [8], the complexity of the list-colouring problem is investigated for planar instances with certain other restrictions, such as list size bounds and degree bounds. In [2] the problem of acyclic digraph homomorphisms was investigated for planar instances.

In this note, we prove the following two theorems.

Theorem 1.1. *For any odd integer $\ell \geq 5$, planar C_ℓ-COL is NP-complete.*

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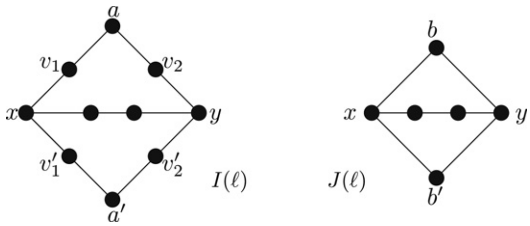

Fig. 1. The graphs $I(5)$ and $J(5)$.

Theorem 1.2. *Let H be any graph of girth* 5 *and maximum degree* 3*, then planar H-COL is NP-complete.*

Theorem 1.1 has recently been proved independently in [5], using a different proof. Our Theorem 1.2 is of interest, as it shows that planar H-COL is NP-complete for many non-planar graphs H. For example, it implies that planar H-COL is NP-complete when H is the Petersen graph.

Outline of the paper. In Section 2 we prove Theorem 1.1. In the other three sections we prove Theorem 1.2. In Section 3 we provide some reductions which will allow us to assume certain useful properties about the graphs H of Theorem 1.2. In Section 4 we use the assumptions about H to provide a structural description of such graphs. In Section 5, we prove that planar H-COL is NP-complete for any graph H satisfying this structural description.

Notation. Throughout the paper, when we consider a C_ℓ-colouring of a graph, we will assume that the vertices of C_ℓ are $[\ell]$, the integers modulo ℓ, and that consecutive integers are adjacent. The same will be true for K_3-colourings. For any set S, an S-colouring is a homorphism to the complete graph having S as its vertex set. For any integer ℓ, an ℓ-path is a path consisting of ℓ vertices. It will often be necessary to consider a fixed embedding of a planar graph G, in this case we will refer to G as a *plane* graph. It is important to note that given a planar graph, a plane embedding can be found in polynomial time. See, for example, [1].

2. Proof of Theorem 1.1

We prove Theorem 1.1 with an indicator-type construction, similar to the constructions of [7]. Given a planar graph G we will construct a planar graph ${}^*G$, and show that

$$ {}^*G \to C_\ell \iff G \to K_3. $$

Because planar K_3-COL is NP-complete, this implies the theorem.

To simplify notation, we assume G has no pendant edges. Clearly, removing pendant edges has no effect on whether or not G has a K_3-colouring.

Before we get to Construction 2.3, which will give us ${}^*G$, we define two graphs that we use in this, and other, constructions.

Definition 2.1 (*The graphs $I(\ell)$ and $J(\ell)$*). Given an odd integer $\ell \geq 5$, define the graphs $I(\ell)$ and $J(\ell)$ as follows.

- The graph $I(\ell)$ is an $(\ell + 2)$-cycle containing the path xv_1av_2y, plus the new path $xv_1'a'v_2'y$, introduced between the vertices x and y.
- The graph $J(\ell)$ is an ℓ-cycle containing the path xby, plus a new vertex b' which is joined to x and y.

The graphs $I(5)$ and $J(5)$ are pictured in Fig. 1.

We will often use $I(\ell)$ and $J(\ell)$ in constructions in the following way.

Definition 2.2. Given a graph with vertices u and v we take a copy of $I = I(\ell)$ and identify u and v with the copies of x and y, respectively. When we do this, we say that we **connect the vertices u and v with a copy of I**.

Similarly when we identify the vertices u and v of a graph G with the copies of b and b' in a copy of $J = J(\ell)$, we say that we **connect the vertices u and v with a copy of J**.

Observe the following properties of $I = I(\ell)$ and $J = J(\ell)$.

P_I: A mapping f of $\{a, a', x, y\} \subset V(I)$ to $[\ell]$, can be extended to a C_ℓ-colouring of I if and only if $f(x)$ and $f(y)$ are distinct elements of the set $\{f(a) - 2, f(a), f(a) + 2\} \cap \{f(a') - 2, f(a'), f(a') + 2\}$.

P_J: J is C_ℓ colourable, and every C_ℓ-colouring assigns b and b' the same colour. (Indeed, this property holds when replacing C_ℓ with any graph containing a copy of C_ℓ and having girth greater than 4.)

Admittedly, P_I is not the most natural characterisation of the colourings of I, but it will prove useful. We now give our indicator-type construction.

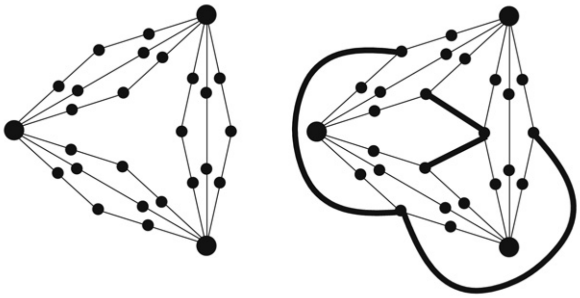

Fig. 2. Steps of Construction 2.3, where $G = K_3$ and $\ell = 5$.

Construction 2.3. *Given an odd integer $\ell \geq 5$ and a plane graph G without pendant edges, construct the plane graph ${}^*G$ from G as follows.*

- *For every edge $e = uv$ of G, remove e and connect u and v with a copy I_e of $I = I(\ell)$. Do so with the embedding of I shown in Fig. 1, so that copies of the vertices a and a' are in different faces of the new graph.*
- *For every face F (including the outer face) of the original graph G, do the following. Let $e_1, e_2, \ldots, e_{|F|}$ denote the edges of the boundary of F, ordered so that consecutive edges share a vertex. Let $a_{F,i}$ be the copy of a or a' in I_{e_i} that lies on the face F. For $i = 1, 2, \ldots, |F| - 1$, connect the vertices $a_{F,i}$ and $a_{F,i+1}$ with a copy $J_{F,i}$ of $J = J(\ell)$.*

See Fig. 2 for an example of this construction. The figure shows the two steps of the construction, as applied to the $G = K_3$ with $\ell = 5$. Thick edges in the figure represent copies of J, with endpoints of the thick edges representing copies of the vertices b and b'.

It is clear that Construction 2.3 can be done so that the resulting graph ${}^*G$ is planar. Furthermore, we use properties P_I and P_J to prove the following.

Lemma 2.4. *For any odd integer $\ell \geq 5$ and any plane graph G without pendant edges, let ${}^*G$ be the planar graph returned by Construction 2.3. Let f be a map from $V(G)$ to $[\ell]$. Then f can be extended to a C_ℓ-colouring of ${}^*G$ if and only if for every face F of G there exists $c_F \in [\ell]$ such that f induces a proper $\{c_F - 2, c_F, c_F + 2\}$-colouring of the boundary of F.*

Proof. Assume that f can be extended to a C_ℓ-colouring, ϕ', of ${}^*G$. For each face F of G, the vertices $a_{F,i}$ are connected by copies of J, thus property P_J implies that $\phi'(a_{F,1}) = \cdots = \phi'(a_{F,|F|})$. Setting $c_F = \phi'(a_{F,1})$, property P_I implies to us that ϕ' induces a $\{c_F - 2, c_F, c_F + 2\}$-colouring of the boundary of F. Thus the same is true of f. This was for a arbitrary face F of G, so we have the forward implication of the lemma.

For the converse implication, let $f : V(G) \to [\ell]$, and let $F \mapsto c_F$ be a mapping of the faces of G to $[\ell]$, such that for every F, f induces a $\{c_F - 2, c_F, c_F + 2\}$-colouring of the boundary of F. We extend f to a C_ℓ-colouring of ${}^*G$ as follows.

For every edge $e = uv$ in G, let F_e and F'_e be the faces of G whose boundaries contain e. Assume, without loss of generality, that the copy of a in I_e lies in F_e, and the copy of a' lies in F'_e. Since u and v are adjacent in F, the definition of the map $F \mapsto c_F$ implies that, $f(u)$ and $f(v)$ are distinct members of both $\{c_{F_e} - 2, c_{F_e}, c_{F_e} + 2\}$ and $\{c_{F'_e} - 2, c_{F'_e}, c_{F'_e} + 2\}$. Thus by property P_I, f can be extended to C_ℓ-colouring of I_e in which $f(a) = c_{F_e}$ and $f(a') = c_{F'_e}$.

Now for any face F of G, we have that $f(a_{F,1}) = \cdots = f(a_{F,|F|}) = c_F$, thus by property P_J, f can be extended to a C_ℓ-colouring of $J_{F,i}$ for $i = 1, \ldots, |F|$. We have thus C_ℓ-coloured ${}^*G$, and so completed the proof of the lemma. $\square$

Now if ϕ is a K_3-colouring of G, then defining $f : V(G) \to [\ell]$ by

$$f(v) = \begin{cases} 1 & \text{if } \phi(v) = 1 \\ 3 & \text{if } \phi(v) = 2 \\ \ell - 1 & \text{if } \phi(v) = 3. \end{cases}$$

Lemma 2.4 implies that there is a C_ℓ-colouring of ${}^*G$ in which $f(a) = f(a') = 1$ for all copies of I. Thus $G \to K_3 \Rightarrow {}^* G \to C_\ell$. We finish the proof of the theorem by proving the following lemma.

Lemma 2.5. *For any odd integer $\ell \geq 5$ and plane graph G, the planar graph ${}^*G$ returned by Construction 2.3 has the property that*

$${}^*G \to C_\ell \Rightarrow G \to K_3.$$

Proof. Towards contradiction, assume that there is some plane graph G for which ${}^*G \to C_\ell$, but $G \not\to K_3$. Further assume that G is a minimum such counter-example with respect to the number of vertices. Let ϕ' be a C_ℓ-colouring of ${}^*G$.

By Lemma 2.4, ϕ' restricts to a 3-colouring (in G) of the boundary of any face of G. We may assume the following.

Claim 2.6. *Every face of G is a triangle.*

Proof. If some face is not a triangle, then $\phi'(u) = \phi'(v)$ for some vertices u and v in the boundary of the face. Let G' be the graph constructed from G by identifying u and v. (Remove any multiple edges thus introduced). Clearly $G' \not\to K_3$.

We now show that there exists a C_ℓ-colouring of ${}^*G'$. Thus G' will be a smaller counter-example to the lemma than G, which will contradict our choice of G. This will prove the claim.

By Lemma 2.4, ϕ' restricts to a map $f : V(G) \to [\ell]$, and induces a map $F \mapsto c_F \in [\ell]$ of the faces of G such that for any face F, f induces a proper $\{c_F - 2, c_F, c_F + 2\}$-colouring of the boundary of F. Since $f(u) = f(v)$, f is a well defined map of $V(G')$. For any face F in G', either the boundary of F is exactly the boundary of a face of G, or it is a path in the boundary of some face of G, in which the endpoints received the same colour under f, and have been identified. In either case, it is clear that f induces on it a $\{c - 2, c, c + 2\}$-colouring for some c in $[\ell]$. Thus by Lemma 2.4, G' has a C_ℓ-colouring.

This proves the claim. □

Now let x be a vertex of G. Since G is a triangulation, the neighbourhood of x is a cycle. For every edge uv in this cycle, Lemma 2.4 implies that $\{\phi'(u), \phi'(v)\}$ is equal to one of $\{\phi'(x) - 4, \phi'(x) - 2\}$, $\{\phi'(x) - 2, \phi'(x) + 2\}$, and $\{\phi'(x) + 2, \phi'(x) + 4\}$. That is to say, the neighbourhood of x in G is a cycle, and admits a homomorphism to the path $\phi'(x) - 4, \phi'(x) - 2, \phi'(x) + 2, \phi'(x) + 4$. Thus the neighbourhood of x in G is an even cycle, and so x has even degree. Since x was an arbitrary vertex of G, the fact that G is 3-colourable follows from the fact that any triangulation in which every vertex has even degree (i.e., any eulerian triangulation) is 3-colourable. This result is apparently from [4], and is given as an exercise (exer. 9.6.2) in [1].

The graph G being 3-colourable contradicts our original assumption, so this completes the proof of the lemma and of the theorem. □

3. Reductions

Our goal in the remainder of the paper is to show that planar H-COL is NP-complete for any graph H of girth 5 and maximum degree 3. Our approach will be to show, through a series of reductions, that any such graph has complexity at least as hard as some graph meeting a certain structural description (Lemma 4.6), and then show that for any graph H meeting this description, planar H-COL is NP-complete. In this section, we give the reductions.

Recall that the *core* of a graph is the unique (up to isomorphism) maximal subgraph whose only monomorphisms are automorphisms. It is well known (see, for example, [6]) that H-COL is polynomially equivalent to C-COL, where C is the core of H.

Thus, in the proof of Theorem 1.2, we will consider only graphs H with the following properties.

(P1) H is a core.
(P2) H has girth 5.
(P3) The maximum degree of H is 3.

In this section we show that we may further assume that H has the following three properties.

(P4) H is connected.
(P5) Every edge of H is in a C_5.
(P6) No vertex v of H, of degree 3, has every pair of neighbours in a C_5.

By the following lemma, H-COL is NP-complete if H'-COL is NP-complete for every component H' of H. Thus we may assume property P4.

Lemma 3.1. *Let H be a fixed graph. Then there is a connected component H' of H such that planar H'-COL can be reduced to planar H-COL in polynomial time.*

Proof. We may assume that H is a core. The lemma is true if H has only one component. Assume that the lemma is true for any graph with $d - 1$ components, for some $d \geq 2$, and let H be a graph with d components. The proof is by induction. Since H has more than one component, and is a core, no component is bipartite.

Let H_1 be a component of H. If every planar graph that admits an H_1-colouring also admits an $(H - H_1)$-colouring, then planar H_1-COL and planar $(H - H_1)$-COL are equivalent, and the lemma follows by induction.

Otherwise, there is some planar graph P that admits an H_1-colouring, but no $(H - H_1)$-colouring. We provide a polynomial time reduction of planar H_1-COL to planar H-COL. (We will often use the letter P for a graph to emphasise the fact that it is planar.)

Since H_1 is not bipartite, it contains an odd cycle. Thus there is some integer k, depending on H_1, such that every (not necessarily distinct) pair of vertices in H_1 is joined by a walk of length k.

Let G_1 be an instance of planar H_1-COL. Construct an instance G of planar H-COL from the disjoint union of P, G_1 and a path of length k (that is, with $k + 1$ vertices) by identifying one end of the path with a vertex of G_1 and the other end of the path with a vertex of P. Then G is a planar graph. Since the graph P depends only on H_1, which is fixed, the transformation can clearly be accomplished in polynomial time.

We claim that there is a homomorphism of G_1 to H_1 if and only if there is a homomorphism of G to H. By definition of k, any homomorphism of G_1 to H_1 extends to a homomorphism of G to H_1, and hence to H. On the other hand, by definition of P, any homomorphism of G to H must map G to H_1. The restriction of this mapping to G_1 is a homomorphism of G_1 to H_1. This completes the proof. □

To justify property P5, construct G' from the given plane graph G as follows. For each edge of G, add a 5-path (involving three new vertices) which joins the endpoints of the original edge. The resulting graph, G', is planar, and has the property that every edge belongs to a 5-cycle. Hence any homomorphism of G' to H must map its subgraph G to the subgraph H' of H induced by the edges of H belonging to 5-cycles. It follows that H'-colouring can be polynomially transformed to H-colouring. This construction can be seen as a special case of the edge-sub-indicator construction from [7] (also see [6]). Hence, we have proved:

Lemma 3.2. *Let H be a graph of odd girth 5, and H' be its subgraph consisting of all edges of that are part of a C_5 in H. Then if planar H'-COL is NP-complete, so is planar H-COL.*

To justify property P6, we uses a simple variation of Construction 2.3.

Lemma 3.3. *Let H be a graph of girth 5 and maximum degree 3. If H has the following property,*

> *There exists a vertex v of degree 3 in H such that for every pair w, w' of neighbours of v there is a C_5 in H containing the path wvw'.*

then planar H-COL is NP-complete.

Proof of Lemma 3.3. Given any (non-empty) plane graph G, construct $^*G$ exactly as in Construction 2.3 for $\ell = 5$, except use copies of $J(5)$ in place of copies of $I(5)$. Thus when we connect vertices u and v of G with a copy of $J(5)$ in the first step of the construction, we will do this by identifying u and v with the copies of x and y of $J(5)$. In the second step, we will be connecting copies of b and b' instead of a and a'.

For every vertex v in a graph H let $\mathcal{N}_v$ be the graph defined by $V(\mathcal{N}_v) = N_H(v)$, and $E(\mathcal{N}_v) = \{uw \mid uvw \text{ is in a } C_5 \text{ in } H\}$. We first prove the following claim.

Claim 3.4. *For any graph H of girth 5,*

$$^*G \to H \iff \exists v \in V(H) \quad \text{such that } G \to \mathcal{N}_v.$$

Proof of claim. Let ϕ be an $\mathcal{N}_v$-colouring of G for some $v \in V(H)$. Define $\phi' : V(^*G) \to V(H)$ as follows. If u is any copy of b or b' in any copy of $J(5)$ in $^*G$, then let $\phi'(u) = v$. If $u \in V(G) \subset V(^*G)$, then let $\phi'(u) = \phi(u)$. One can check that the map ϕ' can be extended to an H-colouring of $^*G$.

On the other hand, let ϕ' be an H-colouring of $^*G$. By property P_J, all copies of b and b' in $^*G$ are mapped to the same vertex by ϕ'. Furthermore, for any edge uv in G, the vertices u and v are neighbours of some copy of b or b' in a copy of C_5 in $^*G$. Thus they must map to distinct neighbours of $\phi'(b)$ in H that are adjacent in $\mathcal{N}_{\phi'(b)}$. That is to say, ϕ' restricted to $V(G)$ is an $\mathcal{N}_{\phi'(b)}$-colouring of G. ◇

The lemma follows. Indeed, since H has maximum degree 3, $\mathcal{N}_v$ is a subgraph of K_3 for every $v \in V(H)$. If H has the property of the lemma, then $\mathcal{N}_v = K_3$ for some v. This allows us to rephrase the biconditional of the claim as

$$^*G \to H \iff G \to K_3.$$

Since $^*G$ is planar if G is, and planar K_3-COL is NP-complete, this gives the conclusion of the lemma. □

4. Structural description of *H*

In this section, we give, with Lemma 4.6, a structural description of the graph H which have the properties P1–P6 listed at the beginning of Section 3. We begin with several definitions which we will need for the statement of Lemma 4.6.

Definition 4.1 (*Ribbons R_n and R_n^t*). For $n \geq 3$, the **n-ribbon**, R_n, is the graph with vertices a_i and b_i for $i = 1, \ldots, n$, and with edges a_ib_i, a_ia_{i+1}, and b_ib_{i+1}, (where indices are mod n) for $i = 1, \ldots, n$. The **twisted n-ribbon**, R_n^t, is R_n with the edges a_nb_1 and b_na_1 instead of a_na_1 and b_nb_1.

The subgraph L_i induced by a_i, a_{i+1}, b_i, and b_{i+1}, is the ith **link** of R_n or R_n^t.

Definition 4.2 (*Subdivided Ribbons S_n and S_n^t*). A **subdivided n-ribbon**, S_n, is any subdivision of R_n in which exactly one edge of every link is subdivided into a 3-path. (See Fig. 3.) S_n^t is an analogous subdivision of R_n^t. We further require that S_n and S_n^t have girth 5 (but this is only a restriction when $n = 3$ or 4).

Every link L_i of S_n or S_n^t thus contains 5 vertices. The fifth vertex, c_i is called the **codicil** of L_i. Such vertices may get more than one label, indeed if the codicil c_i of L_i is introduced in between a_{i+1} and b_{i+1} then it is the codicil of L_{i+1} as well, and so will also be referred to as c_{i+1}.

Observe that many distinct subdivided ribbons are isomorphic. That is, by reindexing the links of a graph S_n or by interchanging the sets $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$, we can get back a different subdivided n-ribbon. There are similar relabellings of S_n^t that allow us to make any link the 'twisted' link. This will often allow us to make certain assumptions about a graph S_n or S_n^t.

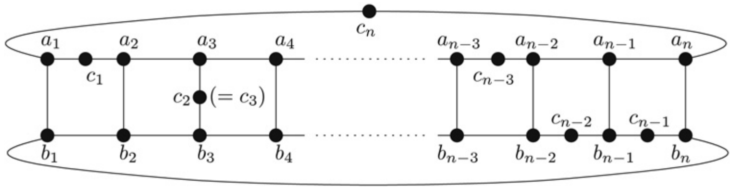

Fig. 3. A sub-divided n-ribbon, S_n.

Definition 4.3 (*Link Types*). The link L_i of a (twisted) subdivided n-ribbon S_n or S_n^t is of type LEFT, TOP, RIGHT, or BOT, if c_i subdivides a_ib_i, a_ia_{i+1}, $a_{i+1}b_{i+1}$, or b_ib_{i+1}, respectively. In the twisted link L_n of S_n^t, types TOP and BOT must be redefined. The link is of type TOP if c_n subdivides a_nb_1, and of type BOT if c_n subdivides b_na_1.

Definition 4.4 (*Broken Ribbons*). A **broken** n-**ribbon**, B_n, is a subgraph of a subdivided n-ribbon S_n with the edges, or 3-paths, between a_n and a_1, and between b_n and b_1, removed.

Observe that a broken n-ribbon has only $n - 1$ links, and up to relabelling, its first and last links can be assumed to be of type LEFT or RIGHT.

Definition 4.5 (*Caps C_A and C_B, and Bridges*). The A-**cap** C_A of a subdivided n-ribbon S_n is the cycle consisting of the vertices a_i, and all codicils of links of type TOP. The B-**cap** C_B of a subdivided n-ribbon S_n is the cycle consisting of the vertices b_i, and all codicils of links of type BOT.

The A-**cap** C_A of a twisted subdivided n-ribbon S_n^t is the cycle consisting of the vertices a_i, all codicils of links of type TOP, b_n, and c_n if L_n is of type BOT. The B-**cap** is defined similarly.

A **bridge** of S_n or S_n^t is a path $c_ix_ic_{i+1}$ with a new vertex x_i, where L_i is of type TOP and L_{i+1} is of type BOT, or vice versa. (When H contains a twisted subdivided n-ribbon S_n^t then we apply this definition to the nth link by moving the twist to some non-adjacent link.)

We can finally give the structural description of a graph H satisfying properties P1–P6. The rest of the section will be dedicated to proving this description.

Lemma 4.6. *Let H be a graph satisfying properties P1–P6. Then H contains a subdivided n-ribbon $S = S_n$ or S_n^t for some $n \geq 4$, with caps of girth at least 6. Its only other vertices or edges are in bridges of S.*

Proof. We begin by establishing the following claim.

Claim 4.7. *H contains a graph B_3.*

Proof of claim. By assumption H contains a copy C of C_5. Since H has maximum degree 3, some vertex of C has a third edge. Since every edge of H is in a C_5, and H has maximum degree three, this third edge is contained in another copy C' of C_5 which intersects C in at least an edge. Using the fact that H has girth 5, and property P6, one can show that C and C' intersect in either an edge (i.e., a 2-path), or a 3-path. That is, H contains one of the two possible broken 3-ribbons B_3. $\diamond$

Let n be the maximum integer for which H contains a broken n-ribbon B_n, and let B a subgraph of H consisting of a copy of B_n and all of its bridges. Furthermore, assume that this B was chosen to maximise the number of bridges. Observe that B is C_5-colourable. Because of this and the assumption that H is a core, H must have some edge not in B. This edge must be in some new C_5, which we call C. Since H has maximum degree 3, C must intersect B in some union of paths each of whose endpoints have degree 2 in B. The following is a list of the ways in which C may intersect B.

(i) One edge (i.e., a 2-path).
(ii) Two independent edges.
(iii) One 3-path.
(iv) An edge and a 3-path.
(v) One 4-path.
(vi) One 5-path.

For each case listed above, we will show that one of the following must be true.

- $B \cup C$ is a (twisted) subdivided n-ribbon in which C is the link L_n.
- One of the properties P1–P6 is contradicted.
- $B \cup C$ is a broken $(n + 1)$-ribbon.
- $B \cup C$ is a broken n-ribbon with more bridges than B.

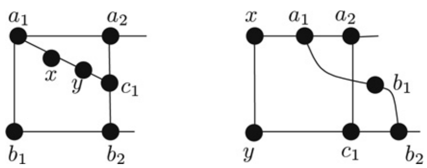

Fig. 4. Redrawing of $B \cup C$ in Case (iii).

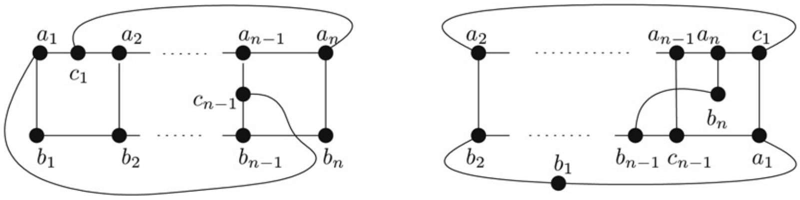

Fig. 5. Redrawing $B \cup C$ in Case (iv).

The lemma follows from any of these conclusions. We now go to the cases.

Case (i) The only places in B where we can have two adjacent degree two vertices, is in L_1 or L_{n-1}. If C intersects B in either one of these places, then $B \cup C$ is a broken $(n+1)$-ribbon.

Case (ii) One of the edges must be in L_1 and one must be in L_{n-1}. The rest of C is an edge and a 3-path between L_1 and L_{n-1}. Thus $B \cup C$ is a (twisted) subdivided n-ribbon.

Case (iii) There are only two places in B where we find two degree two vertices distance two apart.

The first place is the vertices c_i and c_{i+1} when both L_i and L_{i+1} are of type TOP or BOT. If C intersects B in the 3-path between these vertices, then the path's third vertex, a_{i+1} or b_{i+1}, contradicts property P6.

The second place is in L_1 or L_n. Assume, without loss of generality, that C intersects L_1 in a 3-path. We may assume that L_1 is of type LEFT or RIGHT. If L_1 is of type LEFT, then as in Case (ii) we have a broken $(n+1)$-ribbon. If L_1 is of type RIGHT, then, without loss of generality, C is the cycle $a_1a_2c_1yx$, with new vertices x and y. Fig. 4 gives a redrawing of $B \cup C$ in this situation which shows that $B \cup C$ is a broken n-ribbon with one more bridge than B has.

Case (iv) As in Case (iii) we immediately eliminate the possibility that the 3-path has endpoints c_i and c_{i+1}. We may assume then, that the edge is in L_1 and the 3-path is in L_{n-1}. Thus $B \cup C$ is either a (twisted) subdivided n-ribbon, or C is, up to symmetry, one of $a_1c_1a_na_{n-1}c_{n-1}$ (shown in Fig. 5) and $a_1c_1c_{n-1}a_{n-1}a_n$ (not shown). In these last two cases, $B \cup C$ contains S_n and S_n^t respectively. (See Fig. 5 for an example.)

Case (v) If C intersects B in a 4-path, then the endpoints of this path must be vertices of degree two in B, that are exactly distance 3 apart in B, or we will contradict the girth condition. Further, if the 4-path contains a degree 3 vertex a_i, then it must 'turn' at a_i, that is, it must not contain both a_{i-1} (or c_{i-1} if L_{i-1} is of type TOP) and a_{i+1} (or c_i if L_i is of type TOP), or else it contradicts property P6. The same is true if the path contains a degree 3 vertex b_i.

We show that if C intersects B in a 4-path, then either $B \cup C$ is B plus a bridge, or the 4-path contradicts one of these observations.

Consider first the case where both endpoints of the 4-path are codicils. Since they are distance 3 apart, they must be codicils of consecutive links. If these consecutive links L_i and L_{i+1} are one of each of type TOP and BOT, then $B \cup C$ is B plus another bridge. For c_i and c_{i+1} to be distance 3 apart, the only other possibility is that L_i and L_{i+1} are (wlog) of types LEFT and TOP, but then the 4-path is $c_ia_ia_{i+1}c_{i+1}$, which does not 'turn' at a_{i+1}.

We are left with the case where one endpoint is wlog a_1. The only situation in which we cannot relabel vertices of L_1 so that a_1 becomes c_1 (in which case we have already dealt with the case) is if L_1 is of type RIGHT. In this case, the 4-path is either $a_1a_2c_1b_2$ or $a_1b_1b_2c_1$. In the first case we end on a vertex of degree 3, and in the second case, the path $a_1a_2c_1$ shows that the endpoints of the 4-path are distance at most 2 apart.

Case (vi) Similarly to Case (v), C intersects B in a 5-path between vertices of degree 2 that are distance exactly 4 apart in B, and if this path contains a degree 3 vertex a_i or b_i, it must 'turn' at this vertex.

To show this cannot happen, we show by case analysis, that any 5-path between vertices of degree 2 in B which 'turns' at every vertex of degree 3, has endpoints that are distance at most 3 apart.

Again, we may quickly dismiss the case in which the endpoints of the 5-path are codicils. Indeed, assume that c_i, where L_i is of type TOP, is one endpoint. Then wlog, the 5-path begins $c_ia_{i+1}b_{i+1}$. To finish on a vertex of degree two, it must continue $b_{i+2}c_{i+1}$ where L_{i+1} is of type RIGHT, (or if $i+1=n$ then it can finish $c_{i+1}b_{i+2}$, but this is the same up to a relabelling). But here the path $c_ia_{i+1}a_{i+2}c_{i+1}$ shows that the endpoints of the 5-path are distance at most 3 apart.

As in Case (v) we are left with the case where one endpoint is a_1 and L_1 is of type RIGHT. This case is similarly dealt with.

We have thus shown that H must contain S_n or S_n^t, and the only other edges H can have are in bridges. □

This completes the proof of Lemma 4.6.

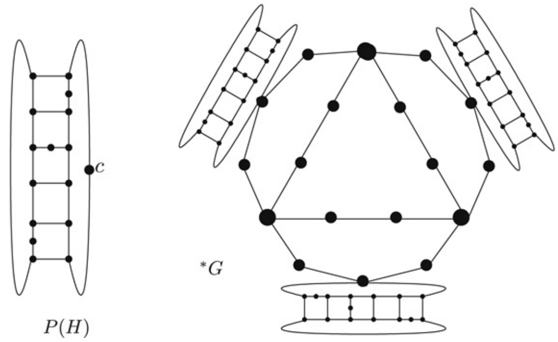

Fig. 6. Construction 5.2 for $G = K_3$.

5. Proof of Theorem 1.2

Let H be any graph of maximum degree 3 and girth 5. To prove Theorem 1.2 we must show that planar H-COL is NP-complete. By the results of Section 3, we may assume that H satisfies the properties P1–P6. Thus H has the structure described in Lemma 4.6.

Lemma 4.6 gives us a structural description of H. However, not all graphs fitting this description are cores. In fact, many graphs fitting this description are C_5-colourable. For those that are cores, we will use an indicator-type construction to show that planar H-COL is NP-complete. When H is planar, the construction is fairly straightforward. When H is not planar, an important part of the construction will be finding a planar subgraph $P(H)$ of H whose possible H-colourings are well understood. We will be able to do this because, in a manner of speaking, the description of H given by Lemma 4.6 is 'almost' planar.

We begin with the following lemma.

Lemma 5.1. *Let H, and its subgraph S, be as in Lemma 4.6, and assume that H is not C_5-colourable. Let c be any vertex in S. Then there exists a planar subgraph $P(H)$ of S, containing c, such that any H-colouring of $P(H)$ is an injection of $P(H)$ to S.*

Proof. Assume that H is not C_5-colourable, and let c be a vertex of the subgraph S of H. Observe that S is not C_5-colourable. Indeed, H is just the union of S with some of its bridges, so a C_5-colouring of S would clearly imply a C_5-colouring of H. We continue the proof in two cases: S is or is not twisted.

In the case that S is a copy of S_n, let $P(H) = S$. We must show that the only H-colourings of S are automorphisms of S.

Let ϕ be an H-colouring of S. For any $i = 1, \ldots, n$, it is easy to see that H is C_5-colourable if we remove at least one edge from each of $C_A \cap L_i$ and $C_B \cap L_i$. Since S is not C_5-colourable, at least one of $C_A \cap L_i$ and $C_B \cap L_i$ is in $\phi(S)$. In either case, $\phi(S)$ contains one of the edges of L_i which is in only one copy of C_5, specifically L_i, in H. Let $\phi(e)$ be this edge. Since e is in a C_5 in S, L_i must be in $\phi(S)$. This is for all i, so S is a subgraph of $\phi(S)$. The lemma follows in this case.

In the case that S is a copy of S_n^t, by reindexing, we may assume that c is not in the nth link. Since S is not C_5 colourable, and $S - (L_n \cap C_A) - (L_n \cap C_B)$ is uniquely C_5-colourable, at least one of the planar graphs $S - (L_n \cap C_A)$ and $S - (L_n \cap C_B)$ is not C_5-colourable. Assume, without loss of generality, that $S - (L_n \cap C_A)$ is not C_5-colourable, and let $P(H) = S - (L_n \cap C_A)$.

By arguments similar to those in the previous case, we can show for any H-colouring ϕ of S, that all links of S except one are in $\phi(S)$, and in the last link, L_i, $\phi(L_n \cap C_A) = L_i \cap C_A$ or $L_i \cap C_B$. Again, the lemma follows.

Now for every plane graph G, we will construct a graph $^*G$ such that

$$^*G \to H \iff G \to K_3.$$

There are several cases to consider. Each case will use the following construction, and will differ only in how we choose the vertex c. In the construction we will use many copies of the cycle C_7. Let a be a vertex of C_7, and x and y be the two vertices that are distance 2 from a.

Construction 5.2. *Given a plane graph G, and a vertex c of the planar subgraph $P(H)$ of H, construct the planar graph $^*G$ as follows.*

- *Replace every edge uv of G with a copy C_{uv} of C_7, by removing the edge uv, and identifying u and v with the copies of x and y in C_{uv}.*
- *For every edge uv of G attach a copy of P_{uv} of $P(H)$ to C_{uv} by identifying the copy of c in P_{uv} with the copy of a in C_{uv}.*

Since G and $P(H)$ are planar, this can be done so that the resulting graph $^*G$ is planar. (See Fig. 6 for an example of this construction when $G = K_3$.)

Now we consider the following cases:

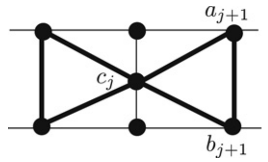

Fig. 7. Case (i) Mapping $^*G$ to H.

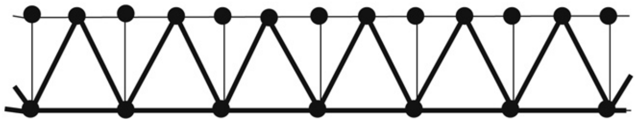

Fig. 8. H^* for Case (ii).

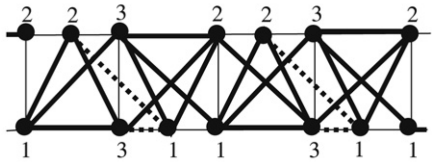

Fig. 9. H^* for Case (iii).

(i) H has a link of type LEFT.
(ii) All links of H are of type TOP or all are of type BOT.
(iii) The links of H alternate between type TOP and BOT.
(iv) The links of H alternate between two of type TOP and two of type BOT.
(v) None of the above hold.

The proof of the backwards implication, $^*G \to H \Leftarrow G \to K_3$, is very similar in each case, thus we prove it only in the first case.

Case (i) Assume that H has a link L_i of type LEFT. Let c be the codicil c_i in the planar subcore $P = P(H)$ of H. Given a planar graph G, let $^*G$ be the graph returned by Construction 5.2 for this choice of P and c. We now show that

$$^*G \to H \iff G \to K_3.$$

Observe that in a subdivided n-ribbon with bridges, the only vertices that are incident to two edges each of whose faces are C_5, are codicils of links of type LEFT or RIGHT. Thus by Lemma 5.1, c must map to one of these codicils under any H-colouring of $P(H)$.

Assume there is an H-colouring ϕ of $^*G$. For every uv in G, the copy of a in the subgraph C_{uv} of $^*G$ is identified with the copy of c in P_{uv}, so $\phi(a)$ must be the codicil of some link of type LEFT or RIGHT. The vertices u and v, which are identified with the copies of x and y in C_{uv}, must then map to the endpoints of one of the thick edges (not necessarily in H) shown in Fig. 7. (We get the same picture on a twisted link.)

The map ϕ thus induces a map ϕ' of G to the graph H^* which is the union of such 'bowties' over all type LEFT links. The graph H^* clearly maps K_3, thus so does G.

On the other hand, assume that we have a homomorphism $\phi' : G \to K_3$. Define $\phi :^* G \to H$ as follows. For every edge e in G, let ϕ take the copy of a in C_e to c_j. For every vertex v of G let $\phi(v)$ be c_j, a_{j+1}, or b_{j+1} if $\phi'(v)$ is 1,2, or 3 respectively. This map can be extended to a homomorphism of $^*G$ to H.

Case (ii) We may assume that all links of H are of type TOP. Let c be any codicil in $P(H)$. Given a planar graph G, let $^*G$ be the graph returned by Construction 5.2. An H-colouring of $^*G$ induces an H^*-colouring of G, where H^* is the graph indicated by the bold edges in Fig. 8. This graph clearly maps to K_3.

Case (iii) Let c be any codicil in $P(H)$. Given a planar graph G, let $^*G$ be the graph returned by Construction 5.2. An H-colouring of $^*G$ induces an H^*-colouring of G, where H^* is (a subgraph of) the graph indicated by the bold edges in Fig. 9. (The broken bold edges will only be in H^* if H contains bridges.) The figure indicates a K_3-colouring of this graph, whether or not the broken edges are present.

Case (iv) One can check that if the links of H alternate between two of type TOP and two of type BOT, then H is C_5-colourable.

Case (v) Recall that the orbit of a vertex v in H is the set $\{u \in V(H) \mid \sigma(v) = u, \sigma \in \mathrm{Aut}(H)\}$. Since the previous four cases cover all cases in which all of the codicils are in the same orbit, we may assume that there are at least two orbits. Furthermore, by eliminating cases (i) and (iv), we may assume that there is a link L_j of type TOP, for which both of the links

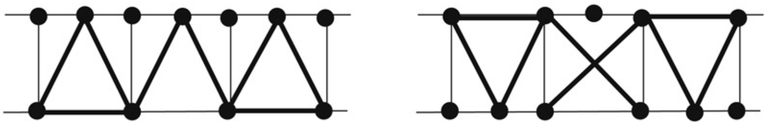

Fig. 10. H^* for Case (v).

L_{j-1} and L_{j+1} are of type TOP or BOT. Let c be some codicil not in the same orbit as c_j. Given a planar graph G, let $^*G$ be the graph returned by Construction 5.2.

An H-colouring of $^*G$ induces an H^*-colouring of G, where H^* is some graph on the vertices of H, which when restriced to the vertices of two consecutive links is some subgraph of the graphs given in Figs. 7–9.

From cases (i), (ii), and (iii), it is clear that we can always 3-colour the subgraph of H^* induced by $H - L_j$. The thick edges shown in Fig. 10 are the only possible edges of the subgraph of H^* that are induced by the vertices of the links L_{j-1}, L_j and L_{j+1}. It follows from this figure that a 3-colouring of the part of H^* induced by the vertices $V(H^*) - V(L_j)$, can be extended to a 3-colouring of H^*.

We have shown that for any graph H meeting the description in Lemma 4.6, either H is C_5-colourable, or Construction 5.2 gives us, for every planar graph G, a planar graph $^*G$ such that

$$^*G \to H \iff G \to K_3.$$

Since planar K_3-COL is NP-complete, so is planar H-COL. This completes the proof of Theorem 1.2.

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